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Author(s): Breeanne Baker Swart and Brittany Shelton
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NOTES

Revelations and Generalizations of the Nine Card Problem

BREEANNE BAKER SWART
Department of Mathematics and Computer Science
The Citadel, The Military College of South Carolina
Charleston, SC 29409
breeanne.swart@citadel.edu

BRITTANY SHELTON
Albright College
Reading, PA 19612
bshelton@albright.edu

Want to see a card trick? Grab a deck of cards; we'll wait. The trick you are about to perform is known as the nine card problem and is credited to magician Jim Steinmeyer [**3**]. The authors first saw this trick performed by Justin Flom on The Ellen Degeneres Show.

Now that you have a deck of cards, pull out any nine cards. Suppose they are A♣ 2♦ 3♠ 4♥ 5♣ 6♦ 7♠ 8♥ 9♣ in this order. Stack (without reshuffling) the cards so that they are face down with the ace of clubs on the top and the nine of clubs on the bottom. Spell out the name of the card that started third from the left, "three of spades." That is,

1. Starting with the top card, spell out "three" by placing one card on the table for each letter. Each new card goes on top of the previous ones. Then place the stack of cards on the table at the bottom of the stack in your hand. The order of the cards is now 6♦ 7♠ 8♥ 9♣ 5♣ 4♥ 3♠ 2♦ A♣ from top to bottom.
2. Do the same for "of." The order of the cards should be 8♥ 9♣ 5♣ 4♥ 3♠ 2♦ A♣ 7♠ 6♦.
3. Repeat for "spades," and the order should be A♣ 7♠ 6♦ 2♦ 3♠ 4♥ 5♣ 9♣ 8♥.

To complete the trick, spell out "Ellen." Here's the kicker: Flip over the last card placed on the table to reveal 3♠!

You are now an amateur magician! To see this trick performed by a professional magician, watch Flom's performance at

http://www.ellentv.com/2013/01/04/card-trick-from-home/.

To add to the performance, audience members, each of whom had a different set of nine cards, performed the trick and "should have" revealed the card he or she spelled—"should have" because not all audience members were successful! In this paper, we use permutations to prove this claim and to generalize the trick.

An explanation of how this trick works as well as suggested ways to spice up its performance can be found in [**2**], where Mulcahy also includes an exercise for the

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reader to prove that the only card whose position is determinable is the third card from the left in a hand of nine cards.

Analysis of the trick

Call the card that starts in the third position from the left the flip card. A simple observation of the position of the flip card after each spelling helps to explain why the trick always works. After one spelling, the flip card will be in the third position from the bottom because when the face-value of every possible card is spelled out, it has at least three letters. The second spelling moves the flip card to the fifth position from the bottom, which is also the fifth position from the top, because "of" has two letters. After the third spelling, the flip card remains in the middle of the stack because each of the four suits has at least five letters. Thus, when the top five cards are placed on the table, the last card to be put on the table is the flip card.

The nine card problem can also be explained using permutations. Before doing so, we introduce some terminology to transition from card tricks to mathematics.

Definition 1. *A* shuffle of length k *is the act of placing* k *cards in reverse order at the bottom of the pile.*

For example, FIGURE 1 shows the original arrangement of a hand of nine cards and the arrangement of the cards after a shuffle of length three.

Each shuffle in the trick is associated to a word, and the length of the shuffle is the number of letters in this word. For example, the shuffle illustrated in Figure 1 is the shuffle associated to the word "ace." It is also the shuffle associated to "two," "six," and "ten" because these words all contain three letters. With this in mind, we construct equivalence classes that partition the set of possible words associated to shuffles. The equivalence classes for the set of possible first shuffles are {ace, two, six, ten}, {four, five, nine, jack, king}, and {three, seven, eight, queen}. There is a single equivalence class for the second shuffle, namely {of}. The equivalence classes for the set of possible third shuffles are {clubs}, {spades, hearts}, and {diamonds}.

Permutations of $[9] = \{1, 2, \ldots, 9\}$ can be used to describe the arrangements of the cards after each possible shuffle. The identity permutation, $12 \cdots 9$, represents the original arrangement of the cards in hand. Define the permutation $w = w_1 w_2 \cdots w_9$ representing the position of the cards after a shuffle by $w_i = j$ if the card originally in position j ends up in position i after the shuffle.

Observe that the shuffle illustrated in FIGURE 1 takes the original arrangement of ace, 2, 3, ..., 9 to the arrangement 4, 5, ..., 9, 3, 2, ace. Thus, the permutation 456789321 represents the arrangement of the cards after the shuffle associated to the

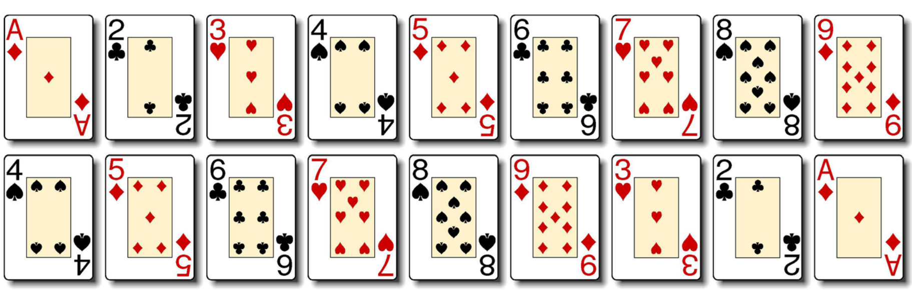

Figure 1 The bottom row shows the rearrangement of the cards in the top row after a shuffle of length three.

equivalence class {ace, two, six, ten}. The shuffles associated to {four, five, nine, jack, king}, {three, seven, eight, queen}, {of}, {clubs}, {spades, hearts}, and {diamonds} correspond to the permutations 567894321, 678954321, 345678921, 678954321, 789654321, and 987654321, respectively.

Successive shuffles can then be represented by composing permutations. The composition of permutation f with permutation g, $f \circ g$, represents the position of each card after shuffles f and g. Keeping in the spirit of composition of functions, we multiply these permutations from right to left. For example, the trick with ace of spades as the flip card is represented by the permutation $456789321 \cdot 345678921 \cdot 789654321 = 154239876$.

Since the card that starts in the third position needs to end in the fifth position for the magic trick to work, this card is not invariant under successive shuffles. In terms of permutations, this means that the permutations representing possible outcomes of the trick have 3 in the fifth position. Thus, neither 3 nor 5 is a fixed point in any of these permutations. However, since 3 is in the same position in each of these permutations, we call it a pseudo fixed point.

Definition 2. *Call $i \in [9]$ a* pseudo fixed point *if each permutation representing the arrangement of the cards after spelling out the name of a card contains i in the same position.*

Using this terminology, we can prove that the card trick always works.

Lemma 1. *For the nine card problem, 3 is the only pseudo fixed point.*

Proof. Notice that each of the permutations that represents a possible first shuffle is of the form $a_1a_2a_3a_4a_5a_6321$. Similarly, each permutation that represents a possible third shuffle is of the form $b_1b_2b_3b_454321$. Because "of" is represented by 345678921, it follows that

$$a_1a_2a_3a_4a_5a_6321 \cdot 345678921 \cdot b_1b_2b_3b_454321$$

represents the outcome of the trick with any first and third shuffles. The resulting permutation has 3 in the fifth position. Thus, 3 is a pseudo fixed point. Because each of the other positions of the resulting permutation depends on the a_i and b_j, the 3 in position 5 is the only pseudo fixed point. ∎

In terms of the card trick, this says that the flip card, the card that starts in the third position, will always be in the fifth position after the third shuffle. Furthermore, the flip card is the only card whose position after all three shuffles is determinable.

Generalizations of the nine card problem

How does the trick change if the words associated to the shuffles are changed? Suppose we spelled out "math for life" instead of the name of a card. The corresponding permutation, 217653498, does not have 3 in position 5. This demonstrates that altering the lengths of the words associated to the shuffles can alter the pseudo fixed points.

First generalization Let's create a new trick by altering the lower bounds on the lengths of the first and third shuffles. Denote the lower bound on the length of the ith shuffle by l_i for $i \in \{1, 3\}$ and the fixed length of the second shuffle by l_2. In the nine card problem, $l_1 = 3$, $l_2 = 2$, and $l_3 = 5$. To further generalize the nine card problem, suppose n cards are held. Now permutations of $[n]$ will be used to represent the arrangement of the cards after each shuffle.

There is a necessary upper bound of n on the length of a shuffle because there are not enough cards to complete a longer shuffle. Furthermore, $l_i > 0$ for all i, otherwise there is no shuffle. Because shuffles of length $n-1$ and n are represented by the same permutation, we do not need to consider the case where $l_i = n$ for any i. Thus, l_1, l_2, and l_3 have values between 1 and $n-1$, inclusive.

The following propositions consider what conditions on the shuffle lengths are required to produce a pseudo fixed point. They also describe the pseudo fixed points in the permutations representing the final positions of the cards.

Proposition 1. *If there exists a pseudo fixed point when three shuffles are performed on n cards, then* $l_1 + l_2 + l_3 \geq n + 1$.

Proof. Suppose $l_1 + l_2 + l_3 \leq n$. After the first shuffle, 1 through l_1 will be in the rightmost l_1 positions in reverse numerical order. Since $l_1 + l_2 + l_3 \leq n$, it follows that $l_2 < n - l_1$. Thus, none of 1 through l_1 are in the leftmost l_2 positions. So after the second shuffle, 1 through l_1 will be in positions $n - l_2 - l_1 + 1$ through $n - l_2$ in reverse numerical order since the leftmost l_2 cards will move to the rightmost positions. Since l_3 is only a lower bound for the number of cards moved, after the third shuffle, the numbers in the $n - l_3$ leftmost positions are unknown. Positions $n - l_3 + 1$ through n contain the numbers from positions 1 through l_3 after the second shuffle, which depend on the length of the first shuffle. Therefore, the number in each position after the third shuffle cannot be determined and there are no pseudo fixed points.

Thus, for there to be a pseudo fixed point, $l_1 + l_2 + l_3 \geq n + 1$. ■

The conditions in Proposition 1 are both necessary and sufficient to determine the existence of a pseudo fixed point, as shown in Proposition 2.

Proposition 2. *If three shuffles are performed on n cards such that* $l_1 + l_2 + l_3 = n + s$, $s \geq 1$, *then the number of pseudo fixed points and their positions depend on the relationship between* l_1, l_2, l_3, *and* s. TABLE 1 *provides the five possible cases for the relationships between these variables, the resulting pseudo fixed points, and their positions.*

TABLE 1: This table summarizes the pseudo fixed points and their positions for the generalized nine card problem.

	$\min\{l_1, l_3, s\}$	Additional Restriction	Pseudo fixed points	Respective positions
1	l_1	$l_2 \leq n - l_1$	$1, 2, \ldots, l_1$	$l_2 + 1, l_2 + 2, \ldots, l_2 + l_1$
2	l_1	$l_2 > n - l_1$	$1, 2, \ldots, l_1$	$l_2 + 1, l_2 + 2, \ldots, n,$ $l_2, l_2 - 1, \ldots, n - l_1 + 1$
3	l_3	$l_3 > n - l_2$	$1, 2, \ldots, l_3$	$l_2 + 1, l_2 + 2, \ldots, n,$ $l_2, l_2 - 1, \ldots, n - l_3 + 1$
4	l_3	$l_3 \leq n - l_2$	$l_1 - s + 1, l_1 - s + 2,$ $\ldots, n - l_2$	$n - l_3 + 1, \ldots,$ $n - 1, n$
5	s		$l_1 - s + 1, \ldots, l_1$	$l_1 + l_2 - s + 1, \ldots, l_1 + l_2$

Since the same methodology is used to prove each of the five cases, we provide only the proof of case 1. To challenge yourself, try proving cases 2 through 5, then compare to the proofs provided in the supplement [**4**]. Although case 5 does not have an additional condition, the proof still has the same structure as the others.

Proof. Case 1: Suppose $l_2 \leq n - l_1$ and $\min\{l_1, l_3, s\} = l_1$. The permutation after the first shuffle is

$$a_1 a_2 \cdots a_{n-l_1} l_1 \cdots 21$$

where $a_1, \ldots, a_{n-l_1}$ are the numbers $l_1 + 1, \ldots, n$. Since l_1 is only a lower bound for the length of the first shuffle, the arrangement of $l_1 + 1$ through n depends on the actual length of the shuffle performed. The second shuffle moves a_1 through a_{l_2}, the l_2 leftmost numbers after the first shuffle, to the rightmost positions. Furthermore, now 1 through l_1 will be in positions $n - l_2$ through $n - l_1 - l_2$, respectively. Therefore, the permutation after two shuffles is

$$a_{l_2+1} \cdots a_{n-l_1} l_1 \cdots 21 a_{l_2} \cdots a_2 a_1.$$

The third shuffle then moves a_{l_2+1} through a_{n-l_1}, 1 through l_1, and a_{l_2} through $a_{l_2-(s-l_1)+1}$, the leftmost l_3 numbers after the second shuffle, to the rightmost positions. Thus, the permutation after three shuffles is

$$c_1 c_2 \cdots c_{n-l_3} a_{l_2-(s-l_1)+1} \cdots a_{l_2} 12 \cdots l_1 a_{n-l_1} \cdots a_{l_2+1}$$

where c_1 through c_{n-l_3} depend on the actual length of the shuffle performed. Therefore, 1 through l_1 are pseudo fixed points and will occur in positions $n - (l_3 - s) - l_1 + 1$ through $n - (l_3 - s)$, respectively, or $l_2 + 1$ through $l_1 + l_2$, respectively. ■

The trick cannot have more than l_1 pseudo fixed points because the cards in the first $n - l_1$ positions are unknown after the first shuffle. Similarly, since the cards in the first $n - l_3$ positions of the arrangement after the third shuffle are unknown, the trick cannot have more than l_3 pseudo fixed points.

Furthermore, a trick cannot have more than $s = n - (l_1 + l_2 + l_3)$ pseudo fixed points because the cards must work their way through the hand and return to the bottom of the stack throughout the three shuffles, which will not occur unless the shuffles have total length more than n.

Corollary 1. *If three shuffles are performed on n cards such that* $l_1 + l_2 + l_3 = n + s$, $s \geq 1$, *there will be* $\min\{l_1, l_3, s\}$ *pseudo fixed points.*

Second generalization The second shuffle in the nine card problem has a fixed length of two. To further generalize, let's consider what happens if the length of the second shuffle can vary just as the first and third shuffles vary. Pseudo fixed points can still occur; however, the conditions that guarantee pseudo fixed points will be different.

Again, $l_i > 0$ for all i and the length of every shuffle has an upper bound of n.

For this generalization, we do not consider the case where $l_i = n - 1$ for any i because the results are the same as when $l_i = n$. Thus, l_1, l_2, and l_3 have values 1 through $n - 2$ inclusive and n.

In order to guarantee that the position of at least one card is determinable, the shuffles must be sufficiently long enough to cycle the cards through the stack. Specifically, $\min\{l_1, l_3\} + l_2$ must be at least $n + 1$ as shown in Proposition 3.

Proposition 3. *If there exists a pseudo fixed point when three shuffles are performed on n cards, then* $\min\{l_1, l_3\} + l_2 \geq n + 1$.

Proof. Suppose $\min\{l_1, l_3\} + l_2 \leq n$. If $\min\{l_1, l_3\} = l_1$, then $l_1 + l_2 \leq n$. The first shuffle will move the leftmost l_1 numbers to the rightmost positions, and the permutation after one shuffle is

$$a_1 a_2 \cdots a_{n-l_1} l_1 \cdots 21$$

where $a_1, \ldots, a_{n-l_1}$ are the numbers $l_1 + 1, \ldots, n$ in an unknown order. Since their order is unknown, $l_1 + 1$ through n cannot be pseudo fixed points.

Since $l_1 + l_2 \leq n$, it follows that $l_2 \leq n - l_1$, and so a_1 , $a_2, \ldots, a_{l_2}$ are moved to the rightmost l_2 positions in reverse order. However, since l_2 is only a lower bound for the length of the second shuffle, 1 through l_1 may or may not be moved to reverse order by the second shuffle. Thus, the positions of 1 through l_1 are also unknown after the second shuffle. So there will not be any pseudo fixed points.

Similarly, if $\min\{l_1, l_3\} = l_3$, then $l_3 + l_2 \leq n$ and no numbers have a determinable position in the permutation representing the composition of the second and third shuffles. Therefore, if a pseudo fixed point exists, then $\min\{l_1, l_3\} + l_2 \geq n + 1$. ■

In addition, the difference between the sum of the lower bounds on two consecutive shuffles (first and second or second and third) and the number of cards determines the number of pseudo fixed points.

Proposition 4. *If* $\min\{l_1, l_3\} + l_2 \geq n + 1$, *then there are* $s = \min\{l_1, l_3\} + l_2 - n$ *pseudo fixed points. Furthermore,* $\min\{l_1, l_3\} - s + 1$ *through* $\min\{l_1, l_3\}$ *are the pseudo fixed points, and they occur in positions* l_2 *through* $l_2 - s + 1$, *respectively.*

Proof. Suppose $\min\{l_1, l_3\} + l_2 \geq n + 1$. Let $s = \min\{l_1, l_3\} + l_2 - n \geq 1$.

The first shuffle moves the leftmost l_1 numbers to the rightmost positions, and the permutation after one shuffle is

$$a_1 a_2 \cdots a_{n-l_1} l_1 \cdots 21$$

where $a_1, \ldots, a_{n-l_1}$ are the numbers $l_1 + 1, \ldots, n$ in an unknown order.

The second shuffle moves the leftmost l_2 numbers to the rightmost positions. Therefore, the permutation after two shuffles is

$$b_1 b_2 \cdots b_{n-l_2} (n - l_2 + 1) \cdots l_1 a_{n-l_1} \cdots a_2 a_1$$

where $b_1, \ldots, b_{n-l_2}$ are $1, \ldots, n - l_2$ in an unknown order.

The third shuffle moves the leftmost l_3 numbers to the rightmost positions. The minimum of l_1 and l_3 determines which numbers are in these l_3 leftmost positions.

Case 1: If $\min\{l_1, l_3\} = l_1$, then $n - l_2 + 1 = l_1 - s + 1$ and $n - l_2 + s = l_1 \leq l_3$. In this case, the leftmost l_3 numbers include b_1 through b_{n-l_2}, $n - l_2 + 1 = l_1 - s + 1$ through l_1, and a_{n-l_1} through a_{n-l_3+1}. Therefore, the permutation after three shuffles is

$$c_1 c_2 \cdots c_{n-l_3} a_{n-l_3+1} \cdots a_{n-l_1} l_1 \cdots (l_1 - s + 1) b_{n-l_2} \cdots b_2 b_1$$

where $c_1, \ldots, c_{n-l_3}$ depend on the actual length of the third shuffle. Thus, $l_1 - s + 1$ through l_1 are pseudo fixed points and occur in positions l_2 through $l_2 - s + 1$, respectively.

Case 2: If $\min\{l_1, l_3\} = l_3$, then $n - l_2 = l_3 - s$. In this case, the leftmost l_3 numbers include b_1 through b_{n-l_2} and $n - l_2 + 1$ through l_3. Therefore, the permutation after three shuffles is

$$c_1 c_2 \cdots c_{n-l_3} l_3 \cdots (n - l_2 + 1) b_{n-l_2} \cdots b_2 b_1$$

where $c_1, \ldots, c_{n-l_3}$ are $l_3 + 1, \ldots, n$ in an unknown order. Thus, $n - l_2 + 1 = l_3 - s + 1$ through l_3 are pseudo fixed points and occur in positions l_2 through $l_2 - s + 1$, respectively. ■

Conclusion

The pseudo fixed points in the permutation associated to the final positions of the cards represent the cards whose positions are determinable after three shuffles. Since 3 is not the only pseudo fixed point in the generalized tricks, the flip card does not have to be the third card from the left. This opens the door to a whole new set of card tricks that you can perform to impress your friends and colleagues.

With the above propositions in hand, we can now return to the exercise posed by Mulcahy in [**2**], which is to show that with n cards, if the card starting in position s always finishes in position t when the words associated to the three shuffles are the three words in the name of a playing card, then $n = 9$, $s = 3$, and $t = 5$. Lemma 1 proves that when nine cards are used, $n = 9$, then necessarily $s = 3$ and $t = 5$.

However, nine is not the only value of n for which the position of a flip card is determinable. By Proposition 1, the number of cards n must be at most nine in order to have a pseudo fixed point because $n < l_1 + l_2 + l_3 = 10$. Since the longest shuffle is of length eight (the shuffle associated to "diamonds"), there must be at least eight cards in hand. Any other construction of such a trick must have $n = 8$. By Proposition 2 Case 5, when $n = 8$, then 2 and 3 are pseudo fixed points that appear in positions 4 and 5, respectively. Thus, $n = 9$, $s = 3$, $t = 5$ is not the only construction of the trick!

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Summary. The nine card problem is a magic trick performed by shuffling nine playing cards according to a set of rules. The magic is that a particular card will always reappear. The success of this trick can be easily explained by considering the lengths of the words in the names of playing cards, which define the shuffling rules. In this paper, we use permutations to prove that the trick will always work. We then use this methodology to generalize the trick to any number of cards with shuffles according to different rules.

BREEANNE BAKER SWART (MR Author ID: 1093737) is an assistant professor of mathematics at The Citadel in Charleston, SC. She received her Ph.D. from Lehigh University under the direction of Garth Isaak. Her research interests include graph theory, combinatorics, and number theory.

BRITTANY SHELTON (MR Author ID: 960329) is an assistant professor of mathematics at Albright College. She earned a B.S. from Montclair State University and a Ph.D. from Lehigh University. Her mathematical interests include algebraic and enumerative combinatorics. She has recently developed an interest in incorporating magic tricks, including this one, into the classroom.